Exact Static and Dynamic Stiffness Matrices for General Variable Cross Section Members

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This paper concerns the formulation of a new finite-element method for the solution of beams with variable cross section. Using only one element, it is possible to derive the exact static and dynamic stiffness matrices (up to the accuracy of the computer) for any polynomial variation of axial, torsional, and bending stiffnesses along the beam. Examples are given for the accuracy and efficiency of the method.

Introduction

THE use of variable cross section members can help the designer reduce the weight of structures. Improved strength and stability can be achieved too. The cost of fabricating such members is relatively high and offsets their advantages. However, when weight and performance are the most important requirements, members with variable cross section are the most suitable choice. One such application might be for the construction of large structures in space.

Many authors have suggested exact and approximate solutions for tapered members. The recent publications by Banerjee and Williams[1,2] gave exact solutions for a few cases of such members. Their solutions are good only for linear taper of the depth, and the cases of I-section and box section with constant width are not included. The variation of the cross-sectional area, polar moment of inertia, and the moment of inertia of the area are not independent, and their variation is dependent on the choice of only two parameters. In the dynamic stiffness matrix, the axial force cannot be accounted for, so that dynamic stability calculations are not possible. Karabalis and Beskos[3] gave exact static stiffness matrices for linear depth variation and constant width. For dynamics and stability, they got good approximate solutions. Eisenberger[4] gave exact static stiffness matrix for beams in bending with linear and parabolic depth variation and constant width, and for linearly varying width with constant depth. Eisenberger and Reich[5] presented an approximate method for general polynomial variation of width and/or depth for beams. All of the above, and many other earlier publications, do not give exact solution for any general polynomial variation of width and depth along the member.

The need for exact solutions is obvious: modeling for tapered members will be simplified, and the solution time cut substantially. Another issue deals with optimization: if one tries to optimize the member dimensions using the currently approximate or exact (limited cases) solutions, the problem becomes large and requires very long solution time, yielding only a local minimum of the objective function. Better solutions will enable much faster optimization with convergence to the global minimum.

The work "exact" in the title means exact up to the accuracy of the computer. It means also that if we solve a particular case using one element, and solve it again using two or more elements, we are going to get the same numerical solution. The same is true for regular prismatic beam elements, where we get the exact solution using any number of elements. For the usual finite-element method, we get better results as we refine the mesh.

In this paper, a new finite-element method[6] is used for variable cross section members, yielding exact results and avoiding the need for mesh refinements and error estimates.

Stiffness Matrix Calculations

The differential equations that govern the displacements (axial, twist, and bending) of a tapered member should be solved in order to obtain the required stiffnesses. A typical member is shown in Fig. 1 with the coordinate system and the numbered degrees of freedom. The stiffness matrix is of size 12×12 for the following degrees of freedom at the two ends of the member: 3 translations and 3 rotations. These are governed by the following differential equations:

1) For axial displacements

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] = \rho A(x)\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $A(x)$ is the cross-sectional area along the beam, E is Young's modulus of elasticity, u is the axial displacement, and ρ is the mass density of the member material.

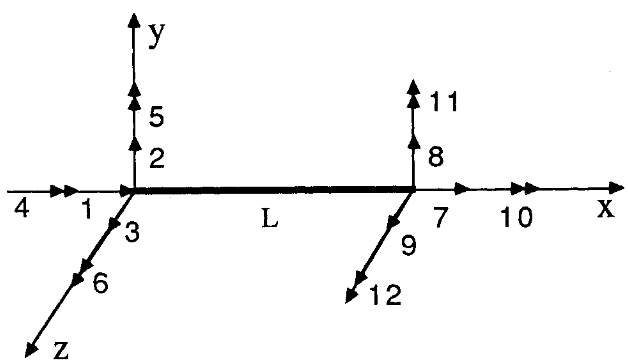

Fig. 1 Typical member and degrees of freedom.

Received Feb. 27, 1989; revision received July 4, 1989.

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2) For torsional rotations

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \theta}{\partial x}\right]=\rho J(x)\frac{\partial^2\theta}{\partial t^2} \tag{2}$$

where $J(x)$ is the polar second moment of inertia, G is the shear modulus, and θ is the angle of twist.

3) For bending displacements (translations and rotations) in the two planes of bending

$$\frac{\partial^2}{\partial x^2}\left[R_v(x)\frac{\partial^2 v}{\partial x^2}\right]=-\rho A(x)\frac{\partial^2 v}{\partial t^2} \tag{3}$$

$$\frac{\partial^2}{\partial x^2}\left[R_w(x)\frac{\partial^2 w}{\partial x^2}\right]=-\rho A(x)\frac{\partial^2 w}{\partial t^2} \tag{4}$$

where $I_v(x)$ and $I_w(x)$ are the moment of inertia along the beam in the two bending planes, v and w are the lateral displacements, and $R_v(x)=EI_v(x)$ and $R_w(x)=EI_w(x)$ for the two directions.

The solution for the general case of polynomial variation of $A(x)$, $J(x)$, $I_v(x)$, and $I_w(x)$ along the beam is not generally available.

In the above expressions, the terms involving derivatives with respect to time represent the inertia forces during vibrations. The dynamic stiffness matrix relates the amplitudes of the sinusoidal varying forces at the ends of the member to the corresponding displacement amplitudes. If we take, for harmonic vibration,

$$Y(x,t)=y(x)\sin\omega t \tag{5}$$

where Y stands for u, θ, v, and w in the equations. Thus, the values of the stiffness terms will depend on ω, the circular frequency of the harmonic vibrations. The static stiffness matrix is evaluated for $\omega=0$, and all of the inertial terms cancel out.

Using the finite-element technique, it is possible to derive the terms in the stiffness matrix. We assume that the shape functions for the element are polynomials and we have to find the appropriate coefficients. It is widely known that exact terms will result if one uses the solution of the differential equation as the shape functions for the derivation of the terms in the stiffness matrix. In this work, "exact" shape functions are used to derive the exact stiffness coefficients. These shape functions are "exact" up to the accuracy of the computer or up to a preset value set by the analyst.

Axial Stiffness

We take the coefficients in Eq. (1) as the following polynomial variation along the beam

$$A(x)=\sum_{i=0}^{l}A_i x^i \tag{6}$$

where l is an integer representing the number of terms in the series. This representation is very general, and many functions can be represented in this way, exactly or up to any desired accuracy.

If we introduce a new local variable ξ

$$\xi=\frac{x}{L} \tag{7}$$

we have for Eqs. (1), (5), and (6)

$$\frac{\partial}{\mathrm{d}\xi}\left[a(\xi)\frac{\mathrm{d}u}{\mathrm{d}\xi}\right]=b(\xi)u \tag{8}$$

$$a(\xi)=\sum_{i=0}^{l}EA_iL^i\xi^i=\sum_{i=0}^{l}a_i\xi^i \tag{9}$$

$$b(\xi)=\sum_{i=0}^{l}\omega^2\rho A_iL^{i+2}\xi^i=\sum_{i=0}^{l}b_i\xi^i \tag{10}$$

Now we choose the solution $u(\xi)$ as the following infinite power series

$$u(\xi)=\sum_{i=0}^{\infty}u_i\xi^i \tag{11}$$

Calculating all of the derivatives and substituting the expressions back into Eq. (8), we have

$$\begin{aligned}0=&\sum_{i=0}^{\infty}\sum_{k=0}^{i}(i-k+1)(i-k+2)a_ku_{i-k+2}\xi^i\\&+\sum_{i=0}^{\infty}\sum_{k=0}^{i}(k+1)(i-k+1)a_{k+1}u_{i-k+1}\xi^i\\&+\sum_{i=0}^{\infty}\sum_{k=0}^{i}b_ku_{i-k}\xi^i\end{aligned} \tag{12}$$

To satisfy this equation for every value of ξ, we must have

$$\begin{aligned}0=&\sum_{k=0}^{i}(i-k+1)(i-k+2)a_ku_{i-k+2}\\&+\sum_{k=0}^{i}(k+1)(\mathrm{i}-k+1)a_{k+1}u_{i-k+1}-\sum_{k=0}^{i}b_ku_{i-k}\end{aligned} \tag{13}$$

or

$$\begin{aligned}u_{i+2}=&\frac{-1}{(i+1)(\mathrm{i}+2)a_0}\left[\sum_{k=0}^{i}b_ku_{i-k}\right.\\&+\sum_{k=0}^{i}(i-k+1)(i-k+2)a_ku_{i-k+2}\\&\left.+\sum_{k=0}^{i}(k+1)(i-k+1)a_{k+1}u_{i-k+1}\right]\end{aligned} \tag{14}$$

so that in Eq. (11) we have all of the u_i coefficients, except for the first two, which should be found using the boundary conditions. The terms for u_{i+2} converge to 0 as $i\to\infty$. As we intend to use this formulation for the derivation of the element stiffness matrix, all of the boundary conditions will be displacements. At $\xi=0$, we have

$$u_0=u(0) \tag{15}$$

so the first term is known from the boundary conditions. The term u_1 is found as follows: all of the u_i are linearly dependent on the first two, and we can write

$$u(1)=\sum_{i=0}^{\infty}u_i=C_0u_0+C_1u_1 \tag{16}$$

The coefficients C_0 and C_1 are functions of all of the coefficients in $a(\xi)$, $b(\xi)$, and ω. C_0, for example, is the value of $u(1)$ when $u_0=1$ and $u_1=0$ calculated from Eq. (11) using the recurrence formula in Eq. (14). In general, we can write all of the C coefficients as follows:

$$C_i=u(\xi=1)=\sum_{k=0}^{\infty}u_k=1+\sum_{k=2}^{\infty}j_k \tag{17}$$

with u_k [from Eq. (14)] based on $u_i=1$, $u_{k\neq i}=0$; $i=0,1$; $k=0,1,2,\ldots,\infty$. Then, knowing all of the terms in Eqs. (16) and (17), the value of u_0 [Eq. (15)] and the boundary conditions at $x=L$ ($\xi=1$), we can solve Eq. (16) and find the unknown u_1. Thus, for any given variable polynomial functions [Eq. (6)], we can find all of the coefficients u_i in Eq. (11). The terms in the stiffness matrix are found in the traditional finite-element method using the following relation:

$$S=\int_0^1 N'^T(\xi)EA(\xi)N'(\xi)\,\mathrm{d}\xi \tag{18}$$

where $N'(\xi)$ are the derivatives of the basis functions. The basis functions (also called shape functions) are found using Eqs. (11), (15), and (16) with boundary conditions as follows:

1) For the first shape function [$N_1(0)=1$; $N_1(1)=0$], take $\mathcal{U}_{0,1}=1$ and then

$$\mathcal{U}_{1,1}=-\frac{C_{0,1}}{C_{1,1}} \tag{19}$$

and then all of the $\mathcal{U}_{i,1}$ are evaluated using Eq. (14).

2) For the second shape function [$N_2(0)=0$; $N_2(1)=1$], take $\mathcal{U}_{0,2}=0$ and then

$$\mathcal{U}_{1,2}=\frac{1}{C_{1,2}} \tag{20}$$

and then all of the $\mathcal{U}_{i,2}$ are evaluated using Eq. (14). The subscripts in the terms for $\mathcal{U}_{i,k}$ in the above equations have the following meaning: i is the term number in the series for shape function number k.

The shape functions that are found using this technique have the special property that they are the "exact" solution for the differential equation. The word exact in the previous sentence stands for "as exact as we can get on a digital computer." This is so since the calculation of the C coefficients is stopped according to a preset criteria: it could be until the contribution of the next element is less than an artibrary small ϵ (in most of the cases ϵ was chosen as 10^{-18}) or until the C values converge completely (for the accuracy of the computer). This property of the shape functions enables us to find the terms in the stiffness matrix in a much faster alternative way [rather than using Eq. (18)] as

$$S(1,1)=-\frac{EA(0)}{L}\mathcal{U}_{1,1} \tag{21}$$

$$S(7,1)=\frac{EA(1)}{L}\sum_{i=0}^{\infty}\mathcal{U}_{i,1} \tag{22}$$

$$S(1,7)=-\frac{EA(0)}{L}\mathcal{U}_{1,2} \tag{23}$$

$$S(7,7)=\frac{EA(1)}{L}\sum_{i=0}^{\infty}\mathcal{U}_{i,2} \tag{24}$$

The terms that are calculated using these equations are exactly equal to the terms that are obtained using Eq. (18). For the calculations, we need only find $S(1,1)$ and then use the equilibrium conditions $S(1,1)=S(7,7)=-S(1,7)=-S(7,1)$.

Torsional Stiffness

The differential equation for the torsion of a member [Eq. (2)] is very similar to the one for axial elongation [Eq. (1)]. If we replace Eqs. (6), (9–11), and (14) with the following equations.

$$J(x)=\sum_{i=0}^{l}J_i x^i \tag{25}$$

$$j(\xi)=\sum_{i=0}^{l}GJ_iL^i\xi^i=\sum_{i=0}^{l}j_i\xi^i \tag{26}$$

$$c(\xi)=\sum_{i=0}^{l}\omega^2\rho J_iL^{i+2}\xi^i=\sum_{i=0}^{l}c_i\xi^i \tag{27}$$

$$\theta(\xi)=\sum_{i=0}^{\infty}\theta_i\xi^i \tag{28}$$

and

$$\theta_{i+2}=\frac{-1}{(i+1)(i+2)j_0}\left[\sum_{k=0}^{i}c_ku_{i-k}+\sum_{k=0}^{i}(i-k+1)(i-k+2)j_k\theta_{i-k+2}+\sum_{k=0}^{i}(k+1)(i-k+1)j_{k+1}\theta_{i-k+1}\right] \tag{29}$$

Then we can derive in a similar way the torsional stiffnesses as

$$S(4,4)=-\frac{GJ(0)}{L}\Theta_{1,1} \tag{30}$$

and $S(4,4)=S(10,10)=-S(4,10)=-S(10,4)$.

Bending Stiffnesses

The bending in the two principal planes is governed by the same differential equation [Eqs. (3) and (4)] with different subscripts. In the following derivation, y will be used for v or w, and the subscripts will be dropped and should be added only to the final formulas for the appropriate stiffness calculations. For bending, the differential equation is a fourth-order equation and the expressions will be longer and more complex. Following the procedure as for the previous two cases, we use the same local variable and polynomial variation of properties along the beam, so that Eqs. (3) or (4) can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[r(\xi)\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right]-\mathrm{d}(\xi)y=0 \tag{31}$$

with

$$r(\xi)=\sum_{i=0}^{n}R_iL^i\xi^i=\sum_{i=0}^{n}r_i\xi^i \tag{32}$$

$$\mathrm{d}(\xi)=\sum_{i=0}^{l}\omega^2\rho A_iL^{i+4}\xi^i=\sum_{i=0}^{l}\mathrm{d}_i\xi^i \tag{33}$$

Now we choose the solution $y(\xi)$ as the following infinite power series:

$$y(\xi)=\sum_{i=0}^{\infty}y_i\xi^i \tag{34}$$

Calculating all of the derivatives and substituting the expressions back into Eq. (31), we have

$$0=\sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{(k+1)(k+2)(i-k+2)!}{(i-k)!}r_{k+2}y_{i-k+2}\xi^i+\sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{2(k+1)(i-k+3)!}{(i-k)!}r_{k+1}y_{i-k+3}\xi^i+\sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{(i-k+4)!}{(i-k)!}r_ky_{i-k+4}\xi^i-\sum_{i=0}^{\infty}\sum_{k=0}^{i}\mathrm{d}_ky_{i-k}\xi^i \tag{35}$$

To satisfy this equation for every value of ξ, we must have

$$0=\sum_{k=0}^{i}\frac{(k+1)(k+2)(i-k+2)!}{(i-k)!}r_{k+2}y_{i-k+2}+\sum_{k=0}^{i}\frac{2(k+1)(i-k+3)!}{(i-k)!}r_{k+1}y_{i-k+3}+\sum_{k=0}^{i}\frac{(i-k+4)!}{(i-k)!}r_ky_{i-k+4}-\sum_{k=0}^{i}\mathrm{d}_ky_{i-k} \tag{36}$$

or

$$y_{i+4}=\frac{1}{(i+1)(i+2)(i+3)(i+4)r_0}\left[\sum_{k=0}^{i}\mathrm{d}_k y_{i-k} - \sum_{k=0}^{i}\frac{(k+1)(k+2)(i-k+2)!}{(i-k)!}r_{k+2}y_{i-k+2} - \sum_{k=0}^{i}\frac{2(k+1)(i-k+3)!}{(i-k)!}r_{k+1}y_{i-k+3} - \sum_{k=1}^{i}\frac{(i-k+4)!}{(i-k)!}r_k y_{i-k+4}\right] \tag{37}$$

and as for the previous derivations, we have all of the y_i coefficients except for the first four, which should be found using the boundary conditions. The terms for y_{i+4} converge to 0 as $i\to\infty$. For this case, we choose as degrees of freedom in the formulation the lateral deflection and rotation at the two ends of the beam element. At $\xi=0$, we have

$$y_0=y(0) \tag{38}$$

and

$$y_1=y'(0) \tag{39}$$

so the first two terms are readily known from the boundary conditions.

The terms y_2 and y_3 are found as follows: all of the y_i are linearly dependent on the first two, and we can write

$$y(1)=\sum_{i=0}^{\infty}y_i=C_0y_0+C_1y_1+C_2y_2+C_3y_3 \tag{40}$$

$$y'(1)=\sum_{i=0}^{\infty}iy_i=C_0'y_0+C_1'y_1+C_2'y_2+C_3'y_3 \tag{41}$$

The eight C coefficients (C_0, C_1, C_2, C_3, C_0', C_1', C_2', and C_3') are functions of all of the coefficients in $r(\xi)$ and ω. C_0, for example, is the value of $y(1)$ when $y_0=1$ and $y_1=y_2=y_3=0$ calculated from Eq. (34) using the recurrence formula in Eq. (37). In general, we can write all of the C coefficients as follows:

$$C_i=y(1)=\sum_{k=0}^{\infty}y_k=1+\sum_{k=4}^{\infty}y_k \tag{42}$$

$$C_i'=y'(1)=\sum_{k=1}^{\infty}ky_k=i+\sum_{k=4}^{\infty}ky_k \tag{43}$$

both with y_k [from Eq. (37)] based on $y_i=1$, $y_{k\neq i}=0$; $i=0,1,2,3$; $k=0,1,2,\ldots,\infty$. Then knowing all of the terms in Eqs. (40-44), the values of y_0 and y_1 [Eqs. (40-41)] and the boundary conditions at $x=L$ ($\xi=1$), we can solve Eqs. (42) and (43) and find the unknowns y_2 and y_3. Thus, for any given variable polynomial functions [Eqs. (32-33)], we can find all of the coefficients y_i in Eq. (37).

As was shown for the axial stiffnesses, there is no need to use the traditional finite-element formulation [Eq. (18)], and the terms in the stiffness matrix can be found directly from the shape functions. The terms in the stiffness matrix are defined as the holding actions at both ends of the beam, due to unit translation or rotation, at each of the four degrees of freedom, one at a time. Thus, there are four sets of boundary conditions as follows:

1) $y(0)=1$	$y'(0)=y(1)=y'(1)=0$
2) $y'(0)=1$	$y(0)=y(1)=y'(1)=0$
3) $y(1)=1$	$y(0)=y'(0)=y'(1)=0$
4) $y'(1)=1$	$y(0)=y'(0)=y(1)=0$

Corresponding to these four sets, there are four solutions $\mathcal{Y}_i$, $i=1,2,3,4$, for $y(\xi)$ which are found using Eqs. (34), (37), and (40-43).

Then, the holding actions will be

$$V(0)=\frac{r(0)}{L^3}\frac{\mathrm{d}^3\mathcal{Y}_i}{\mathrm{d}\xi^3}+\frac{1}{L^2}\frac{\mathrm{d}r(0)}{\mathrm{d}\xi}\frac{\mathrm{d}^2\mathcal{Y}_i}{d\xi^2}=6\frac{R(0)}{L^3}\mathcal{Y}_{i,3}+2\frac{r'(0)}{L^2}\mathcal{Y}_{i,2} \tag{44}$$

$$M(0)=-\frac{r(0)}{L^2}\frac{\mathrm{d}^2\mathcal{Y}_i}{\mathrm{d}\xi^2}=-2\frac{r(0)}{L^2}\mathcal{Y}_{i,2} \tag{45}$$

$$V(1)=-\frac{r(1)}{L^3}\frac{\mathrm{d}^3\mathcal{Y}_i}{\mathrm{d}\xi^3}-\frac{1}{L^2}\frac{\mathrm{d}r(1)}{\mathrm{d}\xi}\frac{\mathrm{d}^2\mathcal{Y}_i}{\mathrm{d}\xi^2}=-\frac{r(1)}{L^3}\sum_{k=3}^{\infty}k(k-1)(k+2)\mathcal{Y}_{i,k}-2\frac{r'(0)}{L^2}\sum_{k=2}^{\infty}k(k-1)\mathcal{Y}_{i,k} \tag{46}$$

$$M(1)=\frac{r(1)}{L^2}\frac{\mathrm{d}^2\mathcal{Y}_i}{\mathrm{d}\xi^2}=\frac{r(1)}{L^3}\sum_{2}^{\infty}k(k-1)\mathcal{Y}_{i,k} \tag{47}$$

where V is the shear force and M is the moment. The stiffnesses for the $u-v$ plane are

$$S(2,i)=6\frac{R_v(0)}{L^3}\mathcal{Y}_{i,3}+2\frac{R_v'(0)}{L^2}\mathcal{Y}_{i,2} \tag{48}$$

$$S(6,i)=-2\frac{R_v(0)}{L^2}\mathcal{Y}_{i,2} \tag{49}$$

$$S(8,i)=-\frac{R_v(1)}{L^3}\sum_{k=3}^{\infty}k(k-1)(k-2)\mathcal{Y}_{i,k}-\frac{R_v'(1)}{L^2}\sum_{k=2}^{\infty}k(k-1)\mathcal{Y}_{i,k} \tag{50}$$

$$S(12,\mathrm{i})=\frac{R_v(1)}{L^2}\sum_{k=2}^{\infty}k(k-1)\mathcal{Y}_{i,k} \tag{51}$$

where $\mathcal{Y}_{i,k}$ are calculated using the r_v coefficients. And the stiffnesses for the $u-w$ plane are

$$S(3,i)=6\frac{R_w(0)}{L^3}\mathcal{Y}_{i,3}+2\frac{R_w'(0)}{L^2}\mathcal{Y}_{i,2} \tag{52}$$

$$S(5,i)=-2\frac{R_w(0)}{L^2}\mathcal{Y}_{i,2} \tag{53}$$

$$S(9,i)=-\frac{R_w(1)}{L^3}\sum_{k=3}^{\infty}k(k-1)(k-2)\mathcal{Y}_{i,k}-\frac{R_w'(1)}{L^2}\sum_{k=2}^{\infty}k(k-1)\mathcal{Y}_{i,k} \tag{54}$$

$$S(11,i)=\frac{R_w(1)}{L^2}\sum_{k=2}^{\infty}k(k-1)\mathcal{Y}_{i,k} \tag{55}$$

were $\mathcal{Y}_{i,k}$ are calculated using the r_w coefficients.

Verification and Examples

The static ($\omega=0$) and dynamic stiffness matrices were tested on numerous examples. When exact solutions were not available, a comparison was made to the solutions obtained from the three approximate methods given by Eisenberger and Reich.[5] In all these cases, all of the results were exactly the same.

The results obtained by this method were compared to those given by Downs[7] for the truncated wedge. All of the results were exactly the same as those given by Downs in his paper

Table 1 Dimensionless frequencies of a truncated wedge cantilever

	Truncation ratio	
	$\alpha = 0.8$	$\alpha = 0.9$
λ_1	3.60828	3.55870
λ_2	20.6210	21.3381
λ_3	56.1923	58.9799
λ_4	109.318	115.187
λ_5	180.163	190.145
λ_6	268.716	283.842
λ_7	374.979	396.278
λ_8	498.950	527.451

(Table 3), and they are not repeated here. Downs had difficulties in calculating the values of the dimensionless frequencies for high taper ratio (defined as α = ratio of tip depth to depth at root for the cantilever with constant width). The values for these two columns in the original table by Downs are given in Table 1, which are exact for all of the figures given.

As an example, a cantilever with variable square cross section was solved. The cantilever was fixed at $\xi = 0$ ($x = 0$). The beam was of unit length, $\rho = 1$, and the variation of the width and depth is given by

$$b(x) = h(x) = 2 - x^2 \tag{56}$$

The static axial stiffness for this member is given as $S(1,1) = 2.46422979453821$. For the example member, the first five axial vibration frequencies were found, using one element, as $\omega_1 = 1.97747$, $\omega_2 = 4.95700$, $\omega_3 = 8.01690$, $\omega_4 = 11.11598$, and $\omega_5 = 14.23223$.

The static lateral stiffnesses for the example member are given as $S(8,8) = S(9,9) = 6.17072668979687$, $S(8,12) = -S(9,11) = 1.64462278352342$, and $S(11,11) = S(12,12) = 0.85745641564910$. For the example member, the first five transverse vibration frequencies were found, using one element, as $\omega_1 = 9.79414$, $\omega_2 = 43.99113$, $\omega_3 = 108.98680$, $\omega_4 = 205.43117$, and $\omega_5 = 333.70723$.

Summary

In this work, exact terms (up to the accuracy of the computer) for the static and dynamic stiffness matrices for any polynomial variation of the properties along the beam are derived. The advantages of the new method are as follows:

1) The shape functions are derived automatically.

2) The shape functions are the exact solutions for the differential equation, and thus, the solution is the exact solution.

3) For continuous variation of member properties, only one element is needed for the exact solution of the unknown function and its derivatives.

4) The exact solution is guaranteed.

5) There is no need for mesh refinements.

The main idea in this new finite-element method is that higher-order elements, in the sense that high-order shape functions are derived, can be used without increasing the number of degrees of freedom. The solution is exact up to any desired accuracy (depending only on the accuracy of the computations on the digital computer).

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